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On Geometry and Sums of Squares

John von Neumann once said, "In mathematics you don't understand things, you just get used to them." The notion of n-dimensional space is now an early entrant in the mathematical curriculum, and few of us view it as particularly mysterious; nevertheless, for generations before ours this was not always the case. To be sure, our experience with the Pythagorean theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$ is easily extrapolated to suggest that for two points $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ in $\mathbb{R}^d$ the distance $\rho(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$ should be given by

$$\rho(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_d - x_d)^2}, \tag{4.1}$$

but, despite the familiarity of this formula, it still keeps some secrets. In particular, many of us may be willing to admit to some uncertainty whether it is best viewed as a theorem or as a definition.

With proper preparation, either point of view may be supported, although the path of least resistance is surely to take the formula for $\rho(\mathbf{x}, \mathbf{y})$ as the *definition* of the Euclidean distance in $\mathbb{R}^d$. Nevertheless, there is a Faustian element to this bargain.

First, this definition makes the Pythagorean theorem into a bland triviality, and we may be saddened to see our much-proved friend treated so shabbily. Second, we need to check that this definition of distance in $\mathbb{R}^d$ meets the minimal standards that one demands of a distance function; in particular, we need to check that ρ satisfies the so-called triangle inequality, although, by a bit of luck, Cauchy's inequality will help us with this task. Third, and finally, we need to test the limits on our intuition. Our experience with $\mathbb{R}^2$ and $\mathbb{R}^3$ is a powerful guide, yet it can also mislead us, and one does well to develop a skeptical attitude about what is obvious and what is not.

Even though it may be a bit like having dessert before having dinner,

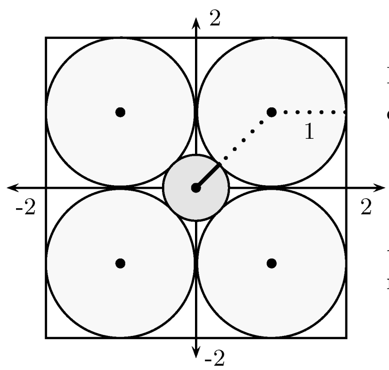

Fig. 4.1. This arrangement of $5 = 2^2 + 1$ circles in $[-2,2]^2$ has a natural generalization to an arrangement of $2^d + 1$ spheres in $[-2,2]^d$. This general arrangement then provokes a question which a practical person might find perplexing — or even silly. Does the central sphere stay inside the box $[-2,2]^d$ for all values of d?

we will begin with the third task. This time the problem that guides us is framed with the help of the arrangement of circles illustrated in Figure 4.1. This simple arrangement of $5 = 2^2 + 1$ circles is not rich enough to suggest any serious questions, but it has a d-dimensional analog which puts our intuition to the test.

On an Arrangement in $\mathbb{R}^d$

Consider the arrangement where for each of the 2^d points denoted by $\mathbf{e} = (e_1, e_2, \ldots, e_d)$ with $e_k = 1$ or $e_k = -1$ for all $1 \le k \le d$, we have a sphere $S_{\mathbf{e}}$ with unit radius and center $\mathbf{e}$. Each of these spheres is contained in the cube $[-2,2]^d$ and, to complete the picture, we place a sphere $\mathcal{S}(d)$ at the origin that has the largest possible radius subject to the constraint that $\mathcal{S}(d)$ does not intersect the interior of any of the initial collection of 2^d unit spheres. We then ask ourselves a question which no normal, sensible person would ever think of asking.

Problem 4.1 (Thinking Outside the Box)

Is the central sphere $\mathcal{S}(d)$ contained in the cube $[-2,2]^d$ for all $d \ge 2$?

Just posing this question provides a warning that we should not trust our intuition here. If we rely purely on our visual imagination, it may even seem silly to suggest that $\mathcal{S}(d)$ might somehow expand beyond the box $[-2,2]^d$. Nevertheless, our visual imagination is largely rooted in